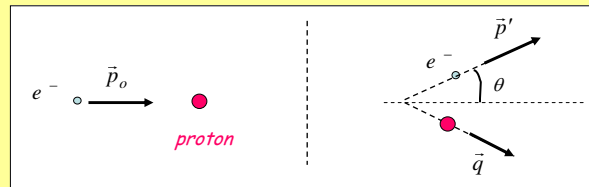Details, Part III: Kinematics

Electrons are **relativistic** → kinetic energy $K \neq p^2/2m$ ...

Einstein mass-energy relations:

(total energy E, rest mass m)

1) $E^2 = (mc^2)^2 + (cp)^2$

2) $E = mc^2 + K$

Problem: units are awkward, too many factors of c ...

Notice that **if c=1**   then (E, m, p, K) all have the same units!

If we set c = 1 in Einstein's mass-energy relations, then in order to "get the answer right", the factor c has to be absorbed in the units of p and m:

1) $E^2 = m^2 + p^2$

2) $E = m + K$

Let the symbol "[   ]" mean "the units of", and then it follows that:

$$[E] = \text{GeV}, \quad [m] = \text{GeV/c}^2, \quad [p] = \text{GeV/c}$$

(Frequently, physicists set c = 1 and quote mass and/or momentum in "GeV" units, as in the graph of the proton electric form factor, lecture 4.   This is just a form of shorthand – they really mean GeV/c for momentum and GeV/c² for mass.... numerically these have the same value because **the value of c is in the unit** – we don't divide by the numerical value 3.00x10⁸ m/s or the answer would be ridiculously small (wrong!)
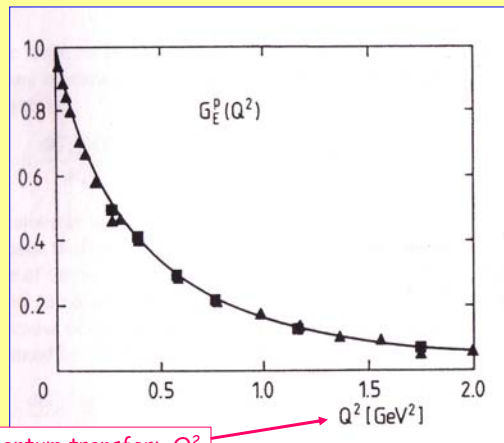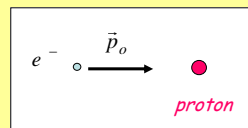
4 – momentum transfer: Q^2

Ref: Arnold et al., Phys. Rev. Lett. 57, 174 (1986)

(Inverse Fourier transform gives the electric charge density $\rho(r)$)

When we want to describe a scattering problem in quantum mechanics, we have to write down wave functions to describe the initial and final states....

For example, the incoming electron is
a free particle of momentum: $\vec{p}_o \ = \ \hbar \, \vec{k}_o$

The electron wave function is:
where V is a normalization volume.
$$\psi(\vec{r}) \ = \ \frac{1}{\sqrt{V}} \ e^{i\vec{k}_o \bullet \vec{r}}$$

If we set $\hbar$ = 1, then momentum p and wave number k have the same units, e.g. fm^{-1};

to convert, use the factor:
$$\hbar c \ = \ 197 \ \text{MeV.fm} \ \rightarrow \ k \ = \frac{cp}{\hbar c}$$

Example:

An electron beam with total energy **E = 5 GeV** has momentum **p = 5 GeV/c** (m ≪ E) ... the same momentum is equivalent to 5 GeV/(0.197 GeV.fm) or **p = k = 25 fm^{-1}**.

So, p = 5 GeV, 5 GeV/c, and 25 fm^{-1} all refer to the same momentum!

Analysis: Kinematics of electron scattering

Note: Elastic scattering is the relevant case for our purposes here.   This means that the beam interacts with the target proton with no internal energy transfer.
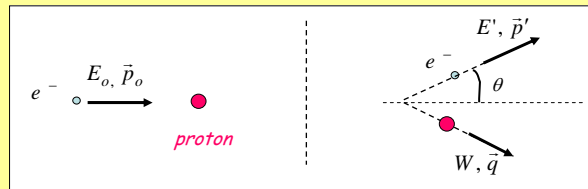
- Specify total energy and momentum for the incoming and outgoing particles as shown.
- Electron mass m << E_o.   Proton mass is M.

Conserve total energy and momentum: $E_o + M \ = \ E' + W, \quad \vec{p}_o = \vec{p}\,' + \vec{q}$

Next steps: find the scattered electron momentum p' in terms of the incident momentum and the scattering angle. Also, find the momentum transfer q^2 as a function of scattering angle, because q^2 will turn out to be an important variable that our analysis of the scattering depends on ...

Details ...

- conserve momentum:

$$\vec{q} \ = \ (\vec{p}_o - \vec{p}\,')$$

1) $q^2 \ = \ p_o^2 + (p')^2 - 2 p_o \ p' \cos\theta$
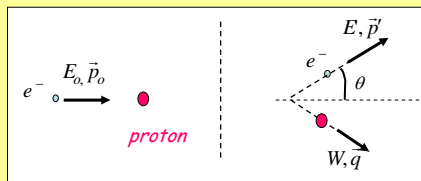
- now use conservation of energy with $W = M + K$ for the proton; $E = p$ for electrons:

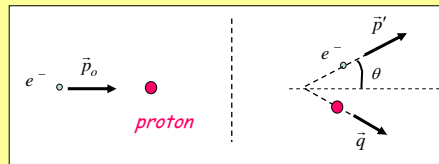$$E_o + M = E' + M + K \quad \Rightarrow \quad K \ = \ (p_o - p')$$

- use kinematic relations to substitute for $K = (p_o - p')$ and q^2:
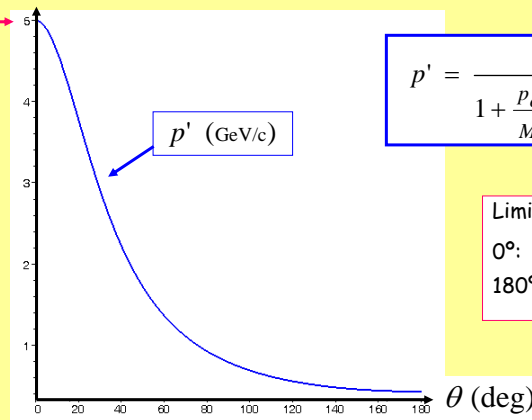
$$W^2 \ \equiv \ M^2 + q^2 \ = \ M^2 + 2MK + K^2$$

Note: these solutions 1), 2) are perfectly general as long as the electron is relativistic. The target can be anything!

2) $p' \ = \ \dfrac{p_o}{1 + \dfrac{p_o}{M}(1 - \cos\theta)}$

Example: 5 GeV electron beam, proton target

$p_o = 5$ GeV/c

$$p' = \frac{p_o}{1 + \frac{p_o}{M}(1 - \cos\theta)}$$

p' (GeV/c)

Limits:

0°:   p' = p$_o$

180°: p'$\cong$ p$_o$/(1+2p$_o$)

θ (deg)

Relativistic 4 - momentum:

It is often convenient to use 4 – vector quantities to work out reaction kinematics.

There are several conventions in this business, all of them giving the same answer but via a slightly different calculation.   We will follow the treatment outlined in Perkins, `Introduction to High Energy Physics', Addison-Wesley (3rd Ed., 1987).

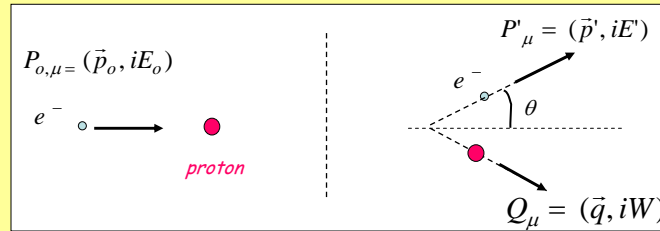Define the relativistic 4-momentum: $P_\mu = (\vec{p},\, iE), \quad \mu = 1...4$

The length of any 4 – vector is the same in all reference frames:

`length' squared $\longrightarrow$ $\sum_\mu P_\mu^2 = p^2 - E^2 = -m^2$

For completeness, a Lorentz boost corresponding to a relative velocity β along the x-axis is accomplished by the 4x4 "rotation" matrix Λ, with:

$\beta = \mathrm{v}/c, \quad \gamma = (1 - \beta^2)^{-1/2}$

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$

$P_{o,\mu} = (\vec{p}_o, iE_o)$

e^-

proton

$P'_\mu = (\vec{p}', iE')$

e^-

θ

$Q_\mu = (\vec{q}, iW)$

Define: 4 - momentum transfer Q between incoming and outgoing **electrons**:

$$Q = (P_o - P') = (\vec{p}_o - \vec{p}', \ i(E_o - E')) = (\vec{q}, i(E_o - E'))$$

Since Q is a 4 – vector, the square of its length is invariant:

$$Q^2 = (\vec{p}_o - \vec{p}')^2 - (E_o - E')^2$$

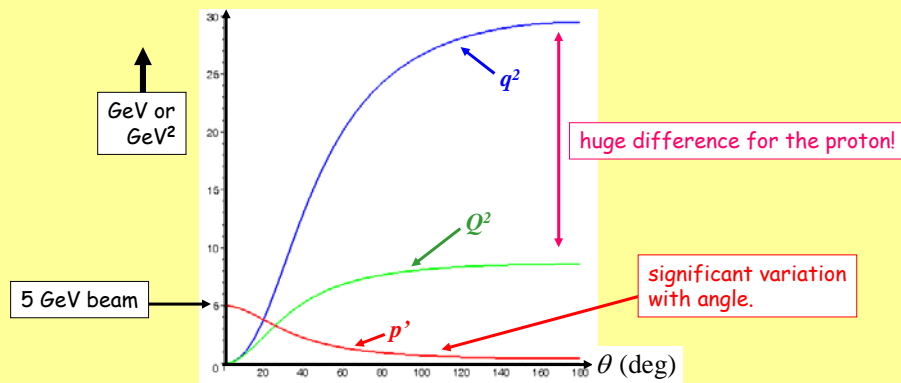Expand, simplify, remembering to use $p^2 - E^2 = -m^2$ and m << E,p ...

$$Q^2 = 2p_o p'(1 - \cos\theta)$$

4–momentum transfer squared
is the variable used to plot high
energy electron scattering data:

$$Q^2 = 2p_o p'(1 - \cos\theta)$$

For a nonrelativistic quantum treatment
of the scattering process (next topic!)
the form factor is expressed in terms of
the **3-momentum transfer** (squared):

$$q^2 = p_o^2 + (p')^2 - 2p_o p' \cos\theta$$

GeV or
GeV²

q^2

huge difference for the proton!

Q^2

significant variation
with angle.

5 GeV beam

p'

θ (deg)

Fraunfelder & Henley use

$$P_\mu = (E, \vec{p}) \;\;\rightarrow\;\; P^2 = E^2 - p^2 = m^2$$

… so with this convention, one has to define a dot product differently, essentially putting in the minus sign by hand.

Our convention is less arbitrary:

$$P_\mu = (\vec{p}, iE) \;\;\rightarrow\; P^2 = p^2 - E^2 = -m^2$$

Note that the norms differ by a minus sign, which carries over to the sign of Q^2

(see the note on p. 139, F&H:

their $Q^2 = -p^2$ at low energy, whereas for us, $Q^2 = +p^2$ in the same regime.

It seems more natural for Q^2 to be a positive number, so our convention is preferred.)

What happens if the target is a nucleus?   (F&H section 6.5)   12

The difference between numerical values of Q^2 and q^2 decreases as the mass of the target increases → we can "get away" with a 3-momentum description (easier) to derive the cross section for scattering from a nucleus.   Note also the simplification that p' ≅ p₀ and becomes essentially independent of θ as the mass of the target increases.   (Why? as M → ∞, the electron beam just 'reflects' off the target – just like an elastic collision of a ping pong ball with the floor – 16.105!!)
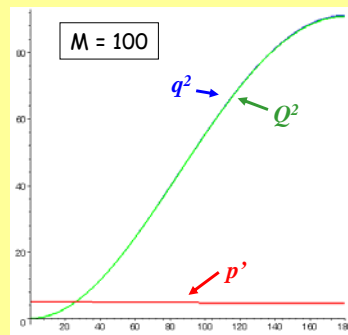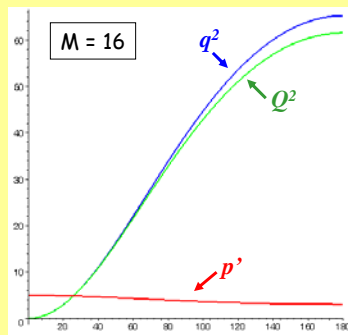
5 GeV electron beam in both cases, as before, but the target mass increased from a proton to a nucleus (e.g., ^{16}O, ^{100}Ru)

Recall from lecture 4:

Before

e^- $\vec{p}_o$

proton

After

detector

$\vec{p}\,'$

e^- θ

$\vec{q}$
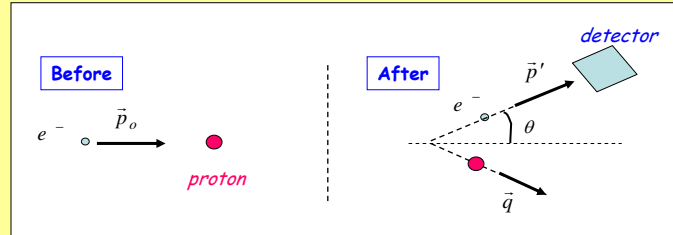
Experimenters detect elastically scattered electrons and measure the cross section:

$$\frac{d\sigma}{d\Omega}(\theta) \;=\; \left.\frac{d\sigma}{d\Omega}\right)_o \; \left[F(q^2)\right]^2$$

point charge result (known)

"Form factor" gives the Fourier transform of the extended target charge distribution. Strictly correct for heavy nuclei: same idea but slightly more complicated expression for the proton...

Last job: we want to work out an expression for the scattering cross section to see how it relates to the structure of the target object.

Scattering formalism: *(nonrel. Q.M.)*   14

e^- e^- θ
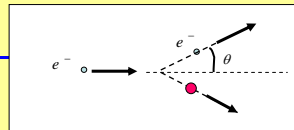
Basic idea:

The scattering process involves a transition between an initial quantum state: $|i$ = incoming e⁻, target p⟩ and a final state $|f$ = scattered e⁻, recoil p ⟩.

The transition rate λ_{if} can be calculated from "Fermi's Golden Rule", a basic prescription in quantum mechanics: (ch. 2)

Units: s⁻¹

$$\lambda_{if} \;=\; \frac{2\pi}{\hbar} \; \left|M_{if}\right|^2 \; \rho_f$$

where the `**matrix element M_{if}**' is given by:

$$M_{if} \;=\; \int \psi_f^* \; V(\vec{r}) \, \psi_i \; d^3r$$

The **potential V(r)** represents the interaction responsible for the transition, in this case electromagnetism (Coulomb's law!),

and the `**density of states**' ρ_f is a measure of the number of equivalent final states per unit energy interval – the more states available at the same energy, the faster the transition occurs.

$$\rho_f \;=\; dn/dE_f$$

Job for next week: relate the transition rate to the scattering cross section!